Problem 4 Illustrating Kronecker products in real life

In practice, you may combine the two small matrices A and B in $A \otimes B$ by sticking B after each ("scalar") element of A, thus obtaining a larger matrix acting on a space of dimensionality equal to the product of the dimensionalities that A and B act on.

For specificity, consider the SU(2) algebra, $[j_x, j_y] = i\ j_z$, $[j_y, j_z] = i\ j_x$ $[j_z, j_x] = i\ j_y$, or, equivalently, for $j_z = j_0$, $j_x = (j_+ + j_-)/\sqrt{2}$, $j_y = -i(j_+ - j_-)/\sqrt{2}$.

Show

$$[j_0, j_+] = j_+ \qquad [j_+, j_-] = j_0 \qquad [j_-, j_0] = j_- \ .$$

Show the (quadratic) Casimir invariant $C \equiv j_x^2 + j_y^2 + j_z^2$ is proportional to $\mathbb{1}$and

$$C = 2j_+j_- + j_0(j_0 - \mathbb{1}) = 2j_-j_+ + j_0(j_0 + \mathbb{1}) \ .$$

For example, skipping the obvious hermitean conjugates, the **2** representation of SU(2) , ($j = 1/2$, Pauli-matrices),

$$j_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad j_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

the **3**,

$$j_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad j_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

the **4**,

$$j_0 = \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix} \qquad j_+ = \begin{pmatrix} 0 & \sqrt{3/2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{3/2} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and so on.

For the addition of angular momenta, two parallel rotations by the same angle on different irreps, say of dimensionalities m and n, tensor-multiply to a rotation generator (spin matrix) operator $\Delta(T_{mn}) \equiv \mathbb{1}_m \otimes T_n + T_m \otimes \mathbb{1}_n$, *satisfying the same SU(2) commutation relations* for SU(2) (show this).

This **coproduct** spin operator is now a reducible representation of SU(2) , the reduction effected by a similarity transformation through the Clebsch-Gordan operator $\mathcal{C}$,

$$\Delta(T_{mn}) \equiv \mathbb{1}_m \otimes T_n + T_m \otimes \mathbb{1}_n = \mathcal{C}(T_1 \oplus T_2 \oplus T_3 \oplus ...)\mathcal{C}^{-1} \ .$$

You worked out in the previous problem the Casimir operator of these general coproduct representations. Does it look messier? (Are there "mixed terms" in the respective vector spaces of dimensions m and n?) But if there were any justice in the world, the same Clebsch operator $\mathcal{C}$ will automatically also reduce this coproduct Casimir to a diagonal matrix, with varying eigenvalues on each of its blocks,

$$\mathcal{C}^{-1} \quad C(\Delta(T)) \quad \mathcal{C} = C_1 \oplus C_2 \oplus C_3 \oplus ...$$

Illustrate the above for the $\mathbf{2}\otimes\mathbf{3}$ case, i.e., adding spin 1/2 to spin 1. Evaluate

$$\Delta(j_0) = \text{diag}\ (3/2, 1/2, -1/2, 1/2, -1/2, -3/2)\ ,$$

$$\Delta(j_+) = \begin{pmatrix} 0 & 1 & 0 & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and reduce by

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2/3} & 0 & -1/\sqrt{3} & 0 & 0 \\ 0 & 0 & 1/\sqrt{3} & 0 & -\sqrt{2/3} & 0 \\ 0 & 1/\sqrt{3} & 0 & \sqrt{2/3} & 0 & 0 \\ 0 & 0 & \sqrt{2/3} & 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(Do you recognize the entries of the orthogonal C-G matrix in the C-G tables of your PDG booklet or website, or favorite QM text?) to $\mathbf{4}\oplus\mathbf{2}$ blocks,

$$\mathcal{C}^{-1}\Delta(J_+)\mathcal{C} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

What is your corresponding Casimir of the coproduct? Is it diagonal? Is it diagonal when likewise reduced by the above C-G matrix?

You might wish to relabel some rows and columns to make the answer look more familiar, or else improve on the above C-G matrix.

Problem 5 The unreasonable power of group theory —SU(2)

Establish the QM van Kortryk identity,

$$e^{\frac{-it}{2\hbar}(\mathbf{p}^2+\mathbf{x}^2)} = e^{\frac{-i}{2\hbar}\mathbf{x}^2\tan(t/2)} e^{\frac{-i}{2\hbar}\mathbf{p}^2\sin t} e^{\frac{-i}{2\hbar}\mathbf{x}^2\tan(t/2)}$$

(which readily produces the quantum harmonic oscillator propagator—the Mehler kernel), relying just on the basic Heisenberg commutator, $[\mathbf{x},\mathbf{p}] = -i\hbar$.

Hint: Examine $[\mathbf{x}^2,\mathbf{p}^2]$. Does it close to a Lie algebra with $\mathbf{p}^2$ and $\mathbf{x}^2$? Can you normalize/recast this algebra to obey the *identical* Lie algebra as the Pauli matrices? If so, you have two different faithful representations of the same Lia algebra—never mind the dimension of this one. As such, all *group element* products must behave identically in each representation (why?); and so it suffices to manipulate the r.h.side product with Pauli matrices instead of the infinite-dimensional operators are considering. You have already encountered Pauli matrix exponentials.

Expand to low order for small t to reassure yourself of the correctness of your expression.

Problem 8

Prove explicitly the isospin current is conserved "on shell", i.e., using the E-L eons of motion. Use the canonical commutation relations of the ϕs and ψs to show the charges

$$Q^a \equiv \int d^3x\ J_0^a(x) = \int d^3x\ \left(-i\phi^*(x)\cdot\tau^a\pi_{\phi^*}(x) + i\phi\cdot\tau^a\pi_\phi - \overline{\psi}(x)\gamma^\mu\cdot\tau^a\psi(x)\right)$$

actually obey the Lie algebra of the relevant isospin symmetry,

$$[Q^a, Q^b] \propto i\epsilon^{abc}Q^c,$$

and further confirm

$$i[Q^a,\phi^i(x)] = ... = \delta\phi^i,$$
$$i[Q^a,\psi^i(x)] = ... = \delta\psi^i,$$
$$i[Q^a,J^b(x)] = ?$$

Moreover, if you were ambitious, work out (sloppily) the equal-time commutators

$$i[J_0^a(x), J_0^b(y)] = ... \qquad @\ t_x = t_y$$

$$i[J_0^a(x), J_\mu^b(y)\phi^i(x)] = ... \qquad @\ t_x = t_y.$$

You might normalize J and Q to conform to our original conventions, if inclined.

Problem 13 Linear $SO(3)$ σ-model
Take three scalars $\phi_{i=1,2,3}$ and write down the three symmetries (w.r.t. three independent angles) of the potential $\lambda(\vec{\phi}^2 - v^2)^2$. Choose $\langle\phi_3\rangle = v$, shift, and write down the linear (unbroken) transformation laws and the nonlinear (shift/ spont.broken) ones. How many Goldstons are there, corresponding to which symmetries?

Generalizing to $SO(N)$ for N fields ϕ_i, what does $SO(N)$ break down to?

Returning to $SO(3)$, introduce *another* scalar triplet $\chi_{i=1,2,3}$, and augment the potential to $\lambda\Big((\vec{\phi}^2 - v^2)^2 + (\vec{\chi}^2 - v'^{\,2})^2 + (\vec{\phi}\cdot\vec{\chi})^2\Big)$. Do any symmetries survive (realized linearly) now?

What combinations of fields are the Goldstone bosons of the spont.broken symmetries? [Hint: Compute the vectors of the 6×6 mass matrix, $\langle\delta\delta V\rangle$.